ON THE NILPOTENT RESIDUE NON-ABELIAN HODGE CORRESPONDENCE FOR HIGHER-DIMENSIONAL QUASIPROJECTIVE VARIETIES

QUOC ANH TRAN

Abstract. In [BBT24], the authors proved that on a projective log smooth variety $(\bar{X}, D)$ there is a continuous bijection between the moduli space $M_{\mathrm{Dol}}^{\mathrm{nilp}}(\bar{X}, D)$ of logarithmic Higgs bundles with nilpotent residues and the moduli space $M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{X}, D)$ of logarithmic connections with nilpotent residues. In this notes, we argue that the map is a homeomorphism.

Let $(\bar{X}, D)$ be a projective log smooth complex variety, i.e., a projective smooth complex variety $\bar{X}$ with a simple normal crossing divisor D. In the sense of [BBT24], we have two good moduli spaces:

$$M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{X}, D, r) = \left\{ (E, \nabla) \;\middle|\; \begin{array}{l} (E, \nabla) \text{ is a rank } r \text{ polystable log connection} \\ \text{with nilpotent residues} \end{array} \right\}$$

$$M_{\mathrm{Dol}}^{\mathrm{nilp}}(\bar{X}, D, r) = \left\{ (E, \theta) \;\middle|\; \begin{array}{l} (E, \theta) \text{ is a rank } r \text{ polystable log Higgs bundle} \\ \text{with nilpotent residues and } c_i(E) = 0 \text{ for all } i \end{array} \right\}$$

and as a corollary of [Moc09, Theorem 1.1], we get a bijective comparison map:

$$SM_{(\bar{X}, D)}^{\mathrm{nilp}} : M_{\mathrm{Dol}}^{\mathrm{nilp}}(\bar{X}, D, r)(\mathbb{C}) \to M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{X}, D, r)(\mathbb{C})$$

which is a homeomorphism, in the Euclidean topology and when $\bar{X}$ is a curve, by [BBT24, Theorem 7.13]. They were also able to prove that $SM_{(\bar{X}, D)}^{\mathrm{nilp}}$ is a continuous bijection in higher dimensions, as follows.

Definition. A Lefschetz curve is a smooth complete intersection curve $\iota : \bar{C} \to \bar{X}$ such that $\bar{C}$ is transverse to D, and we have a surjection $\pi_1(\bar{C} \backslash D) \twoheadrightarrow \pi_1(\bar{X} \backslash D)$.

Such a Lefschetz curve $\iota : \bar{C} \hookrightarrow \bar{X}$ induces a closed embedding $i_{\mathrm{DR}}^* : M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{X}, D, r) \to M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{C}, D \cap \bar{C}, r)$ and a diagram:

$$\begin{tikzcd}
M_{\mathrm{Dol}}^{\mathrm{nilp}}(\bar{X}, D, r) \arrow[r, "i_{\mathrm{Dol}}^*"] \arrow[d, "SM_{(\bar{X}, D)}^{\mathrm{nilp}}"'] & M_{\mathrm{Dol}}^{\mathrm{nilp}}(\bar{C}, D \cap \bar{C}, r) \arrow[d, "SM_{(\bar{C}, D \cap \bar{C})}^{\mathrm{nilp}}"] \\
M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{X}, D, r) \arrow[r, "i_{\mathrm{DR}}^*"'] & M_{\mathrm{DR}}^{\mathrm{nilp}}(\bar{C}, D \cap \bar{C}, r)
\end{tikzcd}$$

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Since $SM^{\text{nilp}}_{(\bar{C},D\cap\bar{C})}$ is a homeomorphism and i^*_{DR} is a closed immersion, $SM^{\text{nilp}}_{(\bar{X},D)}$ is continuous, thus a continuous bijection as desired. Our main result is the following:

Theorem 1. *There exists a Lefschetz curve $\bar{C} \subset \bar{X}$ so that the restriction map i^*_{Dol} is proper.*

Then, by running the previous argument for the inverse comparison maps, we get:

Corollary 2. *$SM^{\text{nilp}}_{(\bar{X},D)}$ is a homeomorphism.*

To get to the proof of theorem 1, we will need some preliminary results and definitions:

Definition. A rank r logarithmic Higgs bundle on $(\bar{X}, D)$ is a rank r locally free $\mathcal{O}_{\bar{X}}-$module E together with a $\mathcal{O}_{\bar{X}}-$linear morphism $\theta : E \to \Omega_{\bar{X}}(\log D) \otimes_{\mathcal{O}_{\bar{X}}} E$ such that $\theta \wedge \theta = 0$, called the Higgs field. Locally, θ is just a matrix of logarithmic 1-forms, hence we can talk about its characteristic polynomial $\text{char}(\theta)$. The coefficients are then in $\bigoplus_{i=0}^r H^0(\bar{X}, \text{Sym}^i \Omega_{\bar{X}}(\log D))$, and we get the Hitchin map:

$$M^{\text{nilp}}_{\text{Dol}}(\bar{X}, D, r) \xrightarrow{h_{(\bar{X},D)}} \bigoplus_{i=0}^{r} H^0\big(\bar{X}, \text{Sym}^i \Omega_{\bar{X}}(\log D)\big)$$

Now if E is only torsion-free, then there is an open $U \subseteq \bar{X}$, with $\text{codim}_{\bar{X}}(\bar{X} - U) \geqslant 2$, such that $E|_U$ is locally free. We can then talk about the characteristic polynomial of $(E|_U, \theta|_U)$. Since $\bar{X}$ is smooth (in particular, normal), the coefficients (over U) extends uniquely to sections over $\bar{X}$. Hence, if we define $M^{\text{tf}}_{\text{Dol}}(\bar{X}, D, P)$ to be the moduli space of Gieseker polystable torsion-free logarithmic Higgs sheaves with Hilbert polynomial P, then we have another Hitchin map with a bigger total space:

$$M^{\text{tf}}_{\text{Dol}}(\bar{X}, D, P) \xrightarrow{h'_{(\bar{X},D,P)}} \bigoplus_{i=0}^{r} H^0\big(\bar{X}, \text{Sym}^i \Omega_{\bar{X}}(\log D)\big)$$

Proposition 3. *[Lan14, Theorem 3.8] The Hitchin map $h_{(\bar{X},D)}$ is proper.*

Proof. Let us say a few words about why this is implied by [Lan14, Theorem 3.8]. That the torsion-free Hitchin map $h'_{(\bar{X},D,P)}$ is proper is just a special case of the cited theorem. Indeed, a logarithmic Higgs sheaf is just a $T_{\bar{X}}(-\log D)-$coHiggs sheaf, viewing $T_{\bar{X}}(-\log D)$ as a trivial Lie algebroid.

$M^{\text{nilp}}_{\text{Dol}}(\bar{X}, D, r)$ is the subscheme of $M^{\text{tf}}_{\text{Dol}}(\bar{X}, D, r \cdot P_{\mathcal{O}_{\bar{X}}})$, where $P_{\mathcal{O}_{\bar{X}}}$ is the Hilbert polynomial of $\mathcal{O}_{\bar{X}}$, cutout by requiring all Chern classes to vanish (which forces the torsion-free sheaves to be locally free) and then requiring the residues of the Higgs fields to be nilpotent. If we can show $M^{\text{nilp}}_{\text{Dol}}(\bar{X}, D, r)$ is a closed subscheme, then $h_{(\bar{X},D)}$, which is the restriction of a proper map to a closed subset, must be proper as desired.

Now, Chern classes are locally constant in a flat family: consider torsion-free $\mathcal{E}$ on $\bar{X} \times S$ with two fibers $\bar{X}_s \xrightarrow{\iota_s} \bar{X} \times S, \bar{X}_t \xrightarrow{\iota_t} \bar{X} \times S$, we claim that if S is connected then $c_i(\iota_s^*\mathcal{E}) \simeq c_i(\iota_t^*\mathcal{E}) \in H^{2i}(\bar{X}, \mathbb{Z})$. By naturality of Chern classes, these are just $\iota_s^* c_i(\mathcal{E})$ and $\iota_t^* c_i(\mathcal{E})$. Since S connected, after identifying $\bar{X}_s \simeq \bar{X}_t \simeq \bar{X}$ it's easy to see ι_s and ι_t are homotopic (via a path γ connecting $s, t \in S$) hence the two pullbacks must agree. Thus requiring vanishing Chern classes pickout a union of connected components of $M^{\text{tf}}_{\text{Dol}}(\bar{X}, D, r \cdot P_{\mathcal{O}_{\bar{X}}})$, hence closed. Further requiring nilpotent residue is also a closed condition, see [BBT24, p. 66]. ∎

Next, we have a commutative diagram:

$$\begin{tikzcd} M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{X}, D, r\big) \arrow[r, "h_{(\bar{X},D)}"] \arrow[d, "i^*_{\mathrm{Dol}}"'] & \bigoplus_{i=0}^{r} H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big) \arrow[d, "\mathrm{res}"] \\ M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{C}, D\cap\bar{C}, r\big) \arrow[r, "h_{(\bar{C},D\cap\bar{C})}"'] & \bigoplus_{i=0}^{r} H^0\big(\bar{C}, \mathrm{Sym}^i\Omega_{\bar{C}}(\log D\cap\bar{C})\big) \end{tikzcd}$$

where the right vertical arrow is the composition

$$H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big) \to H^0\big(\bar{C}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)|_{\bar{C}}\big) \to H^0\big(\bar{C}, \mathrm{Sym}^i\Omega_{\bar{C}}(\log D\cap\bar{C})\big)$$

Proposition 4. *There exists a Lefschetz curve $\bar{C}$ so that* res *is injective.*

Proof. Let's first examine what it means to be in the kernel of the map

$$H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big) \xrightarrow{\mathrm{res}} H^0\big(\bar{C}, \mathrm{Sym}^i\Omega_{\bar{C}}(\log D\cap\bar{C})\big)$$

Suppose that $s \in H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$ maps to the zero section under res. At each point $p \in \bar{C}, p \notin D$, if we restrict s to the fibers at p, we can view $s(p) \in \mathrm{Sym}^i\big(T_p\bar{X}\big)^\vee$ as a homogeneous polynomial on $T_p\bar{X}$. The map res then just restricts $s(p)$ to the line $T_p\bar{C} \subset T_p\bar{X}$. As a result, $\mathrm{res}(s) = 0$ implies that:

$$s(p)\big(T_p\bar{C}\big) = 0 \quad \forall\, p \in \bar{C}, p \notin D$$

Thus it suffices to prove that there exists a Lefschetz curve $\bar{C}$ such that for each $s \in H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$, there exists $p \in \bar{C}, p \notin D$ such that $T_p\bar{C} \not\subset V(s(p)) \subset T_p\bar{X}$. We first prove the following claim:

Claim 5. *There exists a finite number of pairs $\{(p_j, l_j)\}_{j=1}^t$ with $p_j \in \bar{X}, p_j \notin D$, l_j a general 1-dimensional $\mathbb{C}-$subspace of $T_{p_j}\bar{X}$ such that for every $s \in H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$,*

$$s(p_j)(l_j) \neq 0$$

for at least one of the pairs.

Proof of claim. Let $s_1 \in H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$ be a nonzero section, then there is some point $p_1 \in \bar{X}, p_1 \notin D$ such that $s_1(p_1) \in \mathrm{Sym}^i\big(T_{p_1}\bar{X}\big)^\vee$ is nonzero. This is a polynomial on $T_{p_1}\bar{X}$, so the zero locus $V(s_1(p_1))$ is a hypersurface. Thus we can just pick a line l_1 not contained in $V(s_1(p_1))$. The pair (p_1, l_1) gives a linear functional:

$$\epsilon_1 : H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big) \to \mathrm{Sym}^i(l_1^\vee) \simeq \mathbb{C}, \quad s \mapsto s(p_1)|_{l_1}$$

and $\epsilon_1(s_1) \neq 0$. So $\dim\ker(\epsilon_1) < \dim H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$. If $\ker(\epsilon_1) = 0$ we are done, otherwise pick $s_2 \in \ker(\epsilon_1)$ and repeat the process. Since $H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big)$ is finite dimensional, we end up with a finite collection $\epsilon_1, \ldots, \epsilon_t$ such that

$$\bigcap_{j=1}^{t} \ker(\epsilon_j) = 0$$

and the corresponding pairs (p_j, l_j) satisfy the claim. ∎

Our proposition is reduced to showing that there exists a Lefschetz curve $\bar{C}$ which:

- passes through the points $\{p_j\}_{j=1}^t$;
- and for each j, $T_{p_j}\bar{C} = l_j$.

Up to picking a suitable $\mathcal{O}_{\bar{X}}(1)$, this follows from [ACV19, Theorem A.1] part (3) and (4). Indeed, their theorem implies that we can pick a general linear section $\bar{C}$ which is smooth, and passes through $\{p_j\}$ with prescribed tangents. Now, being transverse to D and inducing surjection of fundamental groups are also open conditions (by [HT85, Theorem 1.1.3(ii)]), so there exists a Lefschetz choice. ■

We are finally ready for the main result:

Proof of Theorem 1. We will use the valuative criterion for properness: given a discrete valuation ring R, with $K = \mathrm{Frac}(R)$, and a diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{X}, D, r\big) \arrow[d, "i_{\mathrm{Dol}}^*"'] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{C}, D \cap \bar{C}, r\big)
\end{tikzcd}$$

we want to show that the dotted arrow exists. Consider the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{X}, D, r\big) \arrow[r, "h_{(\bar{X}, D)}"] \arrow[d, "i_{\mathrm{Dol}}^*"'] & \bigoplus_{i=0}^r H^0\big(\bar{X}, \mathrm{Sym}^i\Omega_{\bar{X}}(\log D)\big) \arrow[d, "\mathrm{res}"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] \arrow[urr] & M_{\mathrm{Dol}}^{\mathrm{nilp}}\big(\bar{C}, D \cap \bar{C}, r\big) \arrow[r, "h_{(C, D\cap C)}"'] & \bigoplus_{i=0}^r H^0\big(\bar{C}, \mathrm{Sym}^i\Omega_{\bar{C}}(\log D \cap \bar{C})\big)
\end{tikzcd}$$

The long diagonal arrow exists (and makes the diagram commute) since res is an injective map of vector spaces (equivalently, a closed embedding of affine spaces). Then the dotted arrow must exist since $h_{(\bar{X},D)}$ is proper. Finally, it's a trivial diagram chase to check that the left square commutes. ■

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A. Tran: Dept. of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, USA

Email address: `atran58@uic.edu`